A route to chaos in a nonlinear oscillator with delay

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A Route to Chaos in a Nonlinear Oscillator with Delay

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With 2 Figures

Summary

The paper presents an analysis of the transition from regular to chaotic motion in a Van der Pol-Duffing's oscillator with delay after a Hopf bifurcation. The conditions for the occurrence of the Hopf bifurcation have been determined by means of the approximate method. For the parameters near the bifurcation point a computer simulation of the vibrating system had been performed and the evolution of the system from regular motion to chaos has been analysed at the decrease of the value of the dimensionless damping coefficient.

1. Introduction

Nonlinear oscillators have been examined a long time and it would seem that the knowledge of this subject is complete. Only the works of Ueda and coworkers brought about a return of interest in simple vibrating systems. It is connected with the discovery of irregular (chaotic) motion in such systems. It turned out that chaotic solutions can occur for a forced Van der Pol's oscillator with nonlinear rigidity and for Duffing's oscillator or for other simple physical systems [1]—[5]. They have also been found in nonlinear systems with delay [6]. However, it has been difficult to present a general method to discover them. On the other hand, theories have been formulated, connecting the transition from regular to chaotic motion with the classical theory of dynamical systems. Two basic ones can be distinguished among them. The first one connects the path to chaos through period doubling bifurcations to higher and higher subharmonics as a parameter is varied [7]. The other indicates the possibility of the occurrence of chaos after one, two, or three Hopf bifurcations depending on their mutual position [8]. It appears, however, that the behaviour of real physical systems can often be more complex and does not satisfy the above simplified descriptions. The theories still have to be experimentally confirmed in several cases [9], [10].

This work presents a transition to chaos for the example of a Van der Pol-Duffing oscillator with delay. In this case chaos has been preceded by a Hopf bifurcation of the stationary state.

2. The Analysed System and the Hopf Bifurcation

The equation of motion of the analysed system has the form

$$\ddot{x} - \alpha(1 - x^2)\,\dot{x} + \omega_0{}^2 x + \beta x^3 = kx(t - \tau_0) + F\cos\omega t. \tag{1}$$

It can, for instance, circumscribe the vibrations of the mechanical system presented in [11], where the linear spring force possesses a time delay in its action. As we shall concentrate on the analysis of Eq. (1) for small τ_0 we have

$$x(t - \tau_0) = x(t) - \tau_0\dot{x}(t) + \frac{1}{2}\,\tau_0{}^2\ddot{x}(t). \tag{2}$$

From Eq. (1) we obtain

$$\ddot{x} - (p - \alpha_1 x^2)\,\dot{x} + qx + \beta_1 x^3 = F_1\cos\omega t, \tag{3}$$

where:

$$p = 2(\alpha - k\tau_0)\,(2 - k\tau_0{}^2)^{-1},$$

$$\alpha_1 = 2\alpha(2 - k\tau_0{}^2)^{-1},$$

$$q = 2(\omega_0{}^2 - k)\,(2 - k\tau_0{}^2)^{-1}, \tag{4}$$

$$\beta_1 = 2\beta(2 - k\tau_0{}^2)^{-1},$$

$$F_1 = 2F(2 - k\tau_0{}^2)^{-1}.$$

Equation (3) can be rewritten in the form

$$\dot{x} = z,$$
$$\dot{z} = (p - \alpha_1 x^2)\,\dot{x} - qx - \beta_1 x^3 + F_1\cos\omega t. \tag{5}$$

The approximate solution of (5) is foreseen in the form:

$$x = u\cos\omega t - v\sin\omega t,$$
$$z = -\omega(u\sin\omega t + v\cos\omega t), \tag{6}$$

where u and v are assumed to be slowly varying functions of t. From (6) we have

$$\dot{u}\cos\omega t - \dot{v}\sin\omega t = 0,$$
$$-\omega(\dot{u}\sin\omega t + \dot{v}\cos\omega t) = \dot{z} + \omega^2 x, \tag{7}$$

and, after solving for $\dot{u}$, $\dot{v}$

$$\dot{u} = -\frac{1}{\omega}(z + \omega^2 x)\sin\omega t,$$

$$\dot{v} = -\frac{1}{\omega}(\dot{z} + \omega^2 x)\cos\omega t,$$

(8)

where

$$\dot{z} = \omega^2 x = (p - \alpha_1 x^2)\dot{x} + (\omega^2 - q)x - \beta_1 x^3 + F_1\cos\omega t.$$

(9)

From (8) follows after averaging

$$\dot{u} = \frac{q - \omega^2}{2\omega}v + \frac{pu}{2} - \frac{\alpha_1}{8}u(u^2 + v^2) - \frac{3\beta_1}{8}v(u^2 + v^2),$$

$$\dot{v} = \frac{q - \omega^2}{2\omega}u + \frac{pv}{2} - \frac{\alpha_1}{8}v(u^2 + v^2) + \frac{3\beta_1}{8}u(u^2 + v^2).$$

(10)

Let u_0 and v_0 be a steady state solution of (10), and let $u_1(t)$ and $v_1(t)$ be small disturbances of this solution. After substituting

$$u = u_0 + u_1,$$

$$v = v_0 + v_1,$$

(11)

in (10) and retaining only the terms of first powers of u_1 and v_1 we obtain

$$\dot{u}_1 = Au_1 + Bv_1,$$

$$\dot{v}_1 = Cu_1 + Dv_1,$$

(12)

where:

$$A = \frac{p}{2} - \frac{3}{8}\alpha_1 u_0^2 - \frac{\alpha_1}{8}v_0^2 - \frac{3\beta_1}{4\omega}u_0 v_0,$$

$$B = -\Omega - \frac{\alpha_1}{4}u_0 v_0 - \frac{9}{8}\frac{\beta_1}{\omega}v_0^2 - \frac{3}{8}\frac{\beta_1}{\omega}u_0^2,$$

$$C = \Omega - \frac{\alpha_1}{4}u_0 v_0 + \frac{9}{8}\frac{\beta_1}{\omega}u_0^2 + \frac{3}{8}\frac{\beta_1}{\omega}v_0^2,$$

(13)

$$D = \frac{p}{2} - \frac{\alpha_1}{8}u_0^2 - \frac{3}{8}\alpha_1 v_0^2 + \frac{3}{4}\frac{\beta_1}{\omega}u_0 v_0,$$

$$\Omega = \frac{q - \omega^2}{2\omega}.$$

The necessary conditions for the Hopf bifurcation (see [12]) are:

$$A + D = 0,$$

$$AD - CB > 0.$$

(14)

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The first equation of (14) gives

$$\frac{2p}{\alpha_1} = u_0{}^2 + v_0{}^2,$$ (15)

which is satisfied when $\alpha_1 > k\tau_0$.

Now we introduce polar coordinates $r_0 \in (-\infty, +\infty)$ and $\Theta_0 \in (0, 2\pi)$. Then

$$u_0 = r_0 \cos \Theta_0,$$
$$v_0 = r_0 \sin \Theta_0.$$ (16)

Because u_0 and v_0 satisfy (10), we obtain after substituting (16) into (10)

$$4\omega^2 r_0{}^2 \left[\frac{1}{4}\left(p - \frac{\alpha_1}{4} r_0{}^2\right)^2 + \left(\Omega + \frac{3}{8}\frac{\beta_1}{\omega} r_0{}^2\right)^2\right] = F_1{}^2.$$ (17)

Introducing (15) in (17) gives

$$\frac{1}{2}p^3\left(\alpha_1\omega^2 + \frac{9\beta_1{}^2}{\alpha_1}\right) + 12\beta_1\omega\Omega p^2 + 8p\alpha_1\omega^2\Omega^2 - F_1{}^2\alpha_1{}^2 = 0.$$ (18)

While from the second condition of Eq. (14) we obtain

$$-\frac{1}{16}p^2 + \frac{27}{16}\frac{\beta_1{}^2 p^2}{\omega^2\alpha_1{}^2}p + \Omega^2 > 0.$$ (19)

The parameters of Eq. (3) should satisfy Eq. (18) and Eq. (19) in the critical point. When, additional, the real part of the complex conjugate eigenvalues of Eq. (12) are negative for $\alpha < \alpha_c$ and with the increase of α ($\alpha > \alpha_c$) their real part becomes positive a Hopf bifurcation occurs. In the considered case the critical value of $\alpha_c = .09$ (the other parameter values are: $F = 109.54$, $\omega = 1.6$, $\omega_0 = 1.$, $\beta = 10.$, $k = 5.$, $\tau_0 = .07$).

3. Computer Simulation Results

The numerical simulation of the Eq. (1) is made using the Runge-Kutta method, while the results are presented in the form of the time histories $x(t)$, phase portraits $\dot{x}(x)$, Poincarè maps and frequency spectra. Only the analysis of all four diagrams gives a full view of the behaviour of the system. Poicarè maps are registered after $T_{\min} = 50$. It resulted from the performed numerical calculations that the duration of the transient state, caused by the used initial function is extended with the increase of delay τ_0, hence for small τ_0 a relatively small value of $T_{\min}$ has been assumed. The following initial function is introduced: $x(t) = 1.$ for $-\tau_0 \le t < 0$ and $x(t) = 0.$ for $t = 0.$, as well as $\dot{x}(t) = 0.$ for $-\tau_0 \le t \le 0$.

The results of the numerical simulations are presented in Fig. 1. The calculations are performed with the integration step .01 for arbitraly chosen parameters
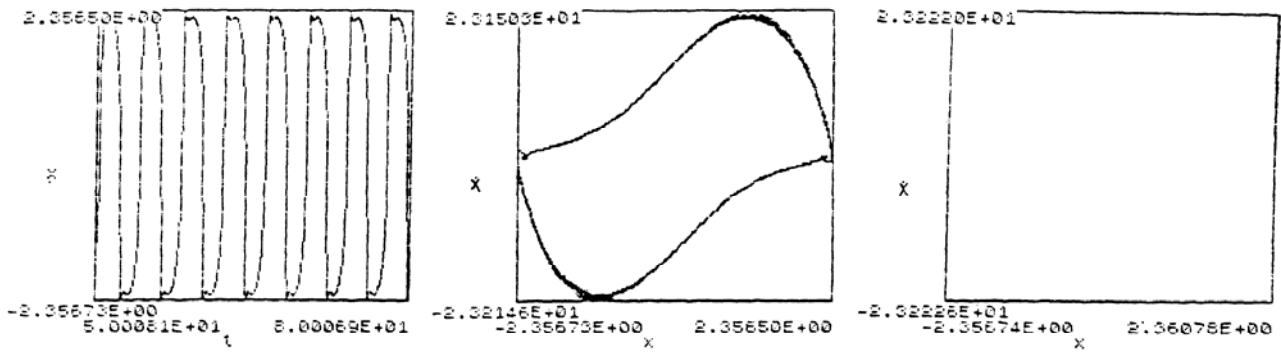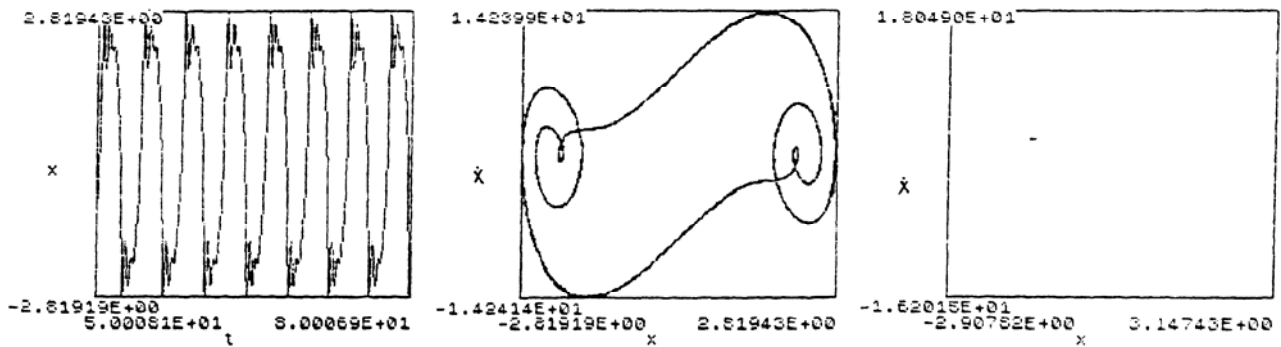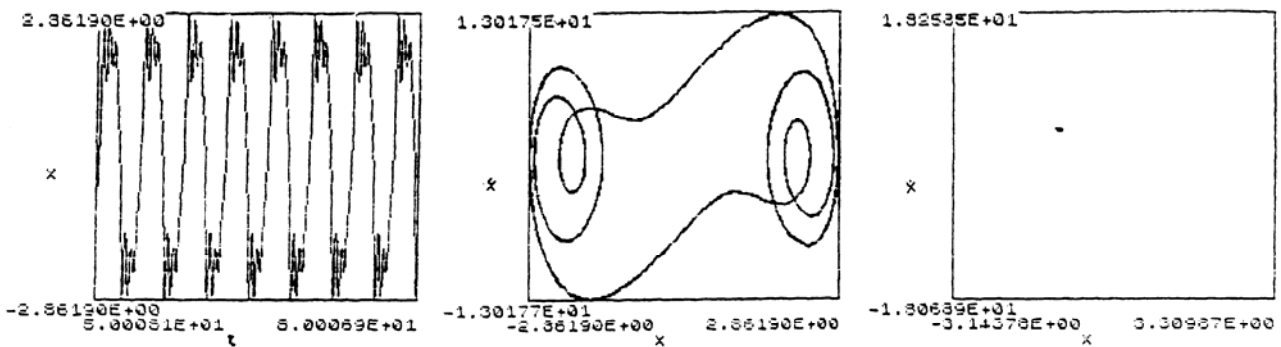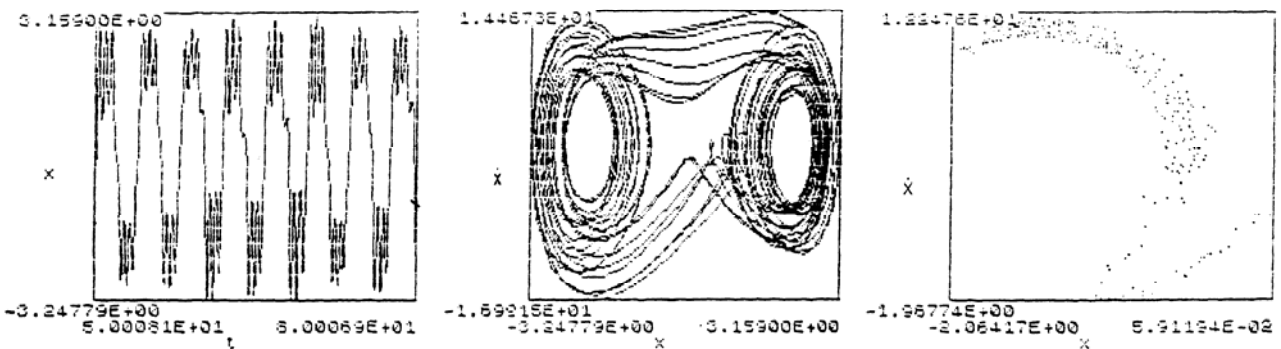
Fig. 1. a)

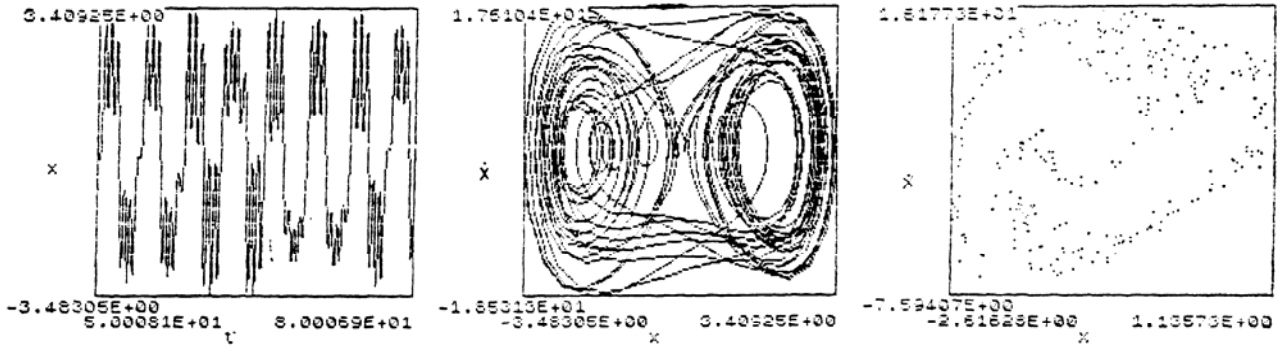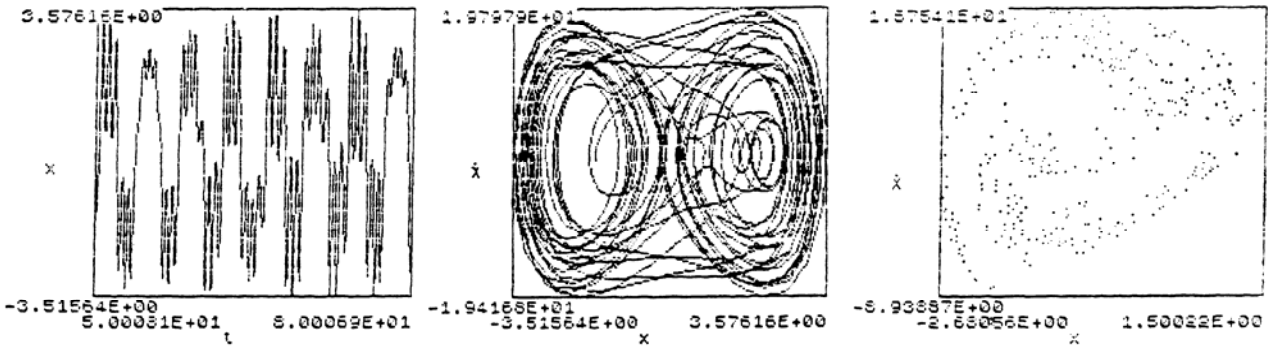
Fig. 1 b)

Fig. 1 c)

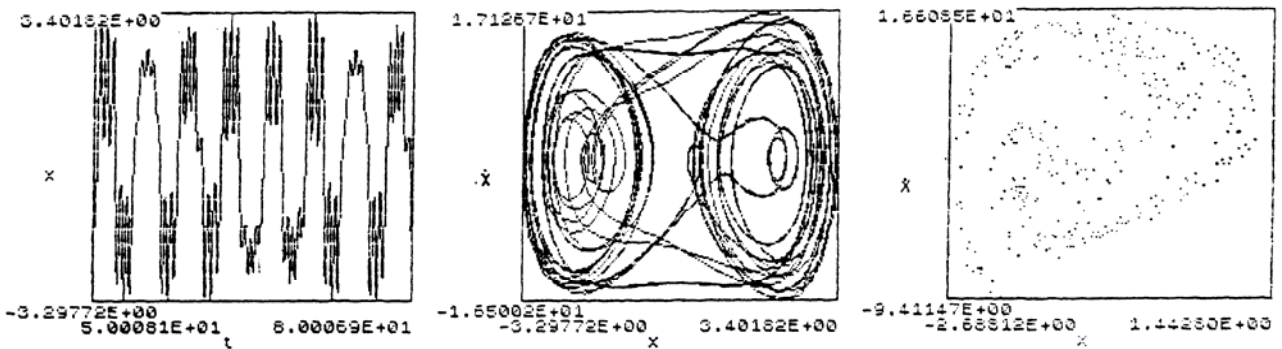Fig. 1 d)

Fig. 1 e)

Fig. 1 f)

Fig. 1 g)

Fig. 1. Time histories, phase portraits and Poincarè maps for α: a) 8., b) 1., c) .5, d) .1,
e) .01, f) .001, g) .0

$F = 109.54$, $\omega = 1.6$, $\omega_0 = 1.$, $\beta = 10.$, $k = 5.$, $\tau_0 = .07$ and Poincarè maps
are made for $T_{max} = 1400$. One can observe the qualitative changes of the solution
before the chaotic regime is reached. For a great damping coefficient $\alpha = 8.$, the
vibrations are periodic but with eight amplitudes of the Fourier components
against frequency, which decrease almost exponentially (Fig. 2). With the decrease
of the value of α (see Fig. 1 b, c and corresponding Fourier spectra) the components
of the solution with the other harmonics increase compared with the basic one.
Moreover, the Fourier spectrum becomes more and more irregular. At $\alpha = .1$
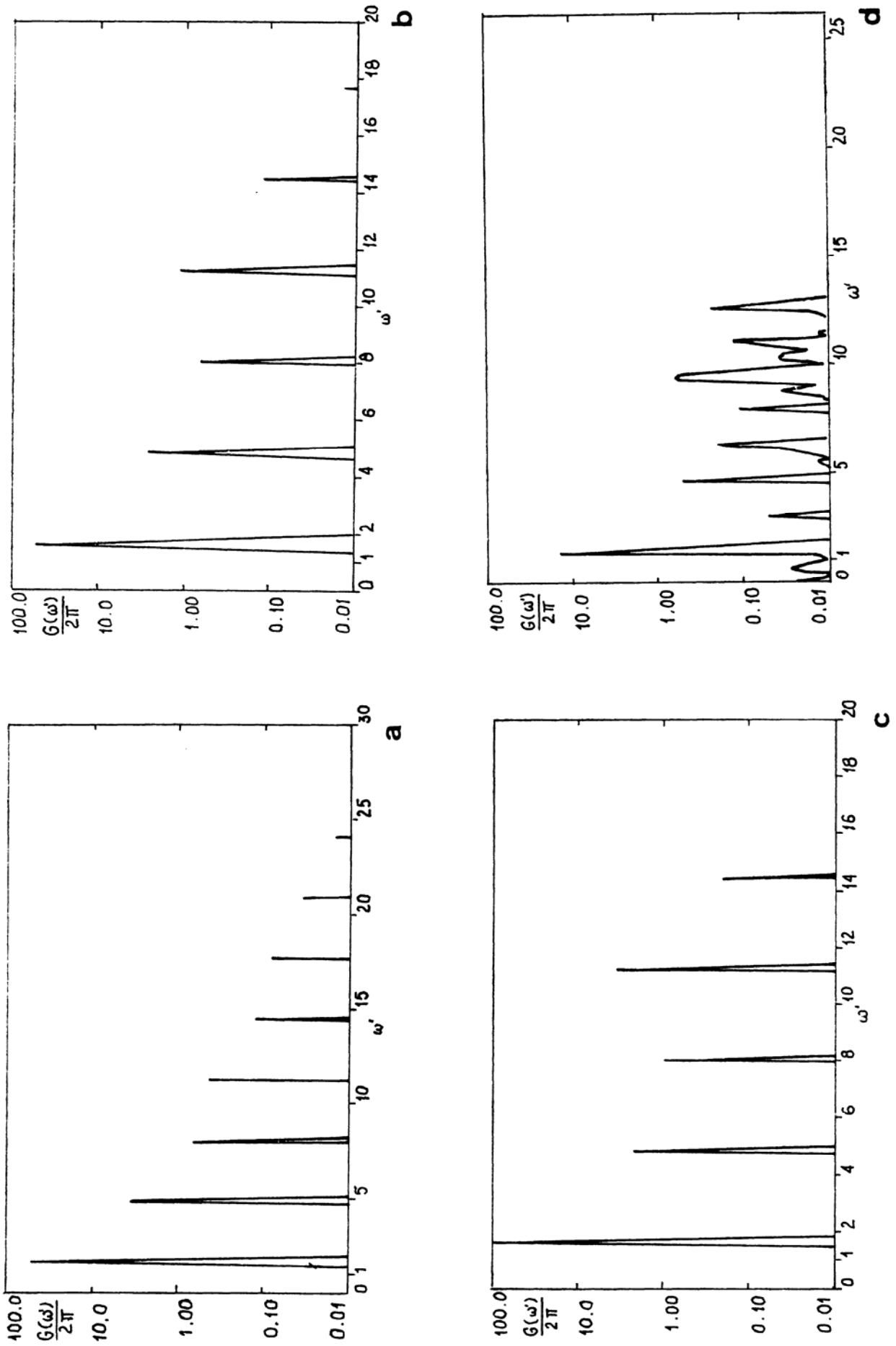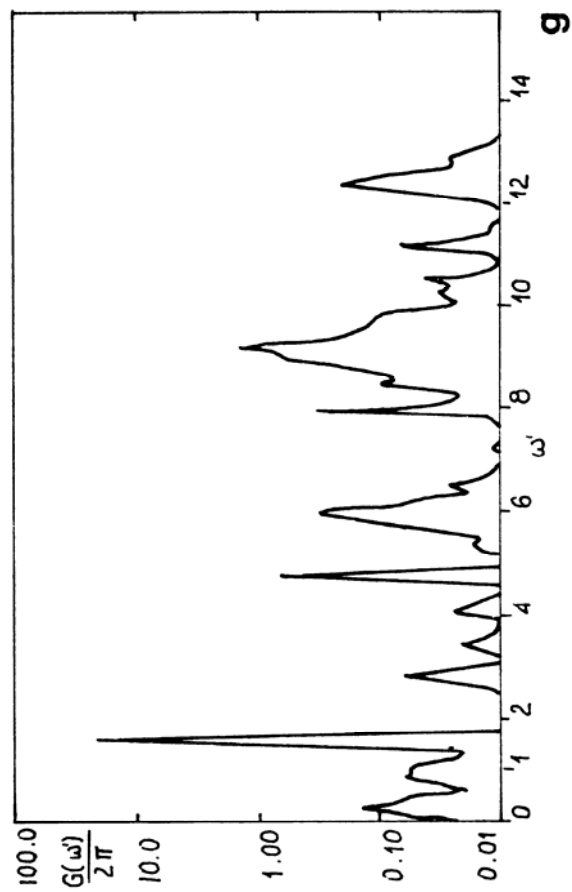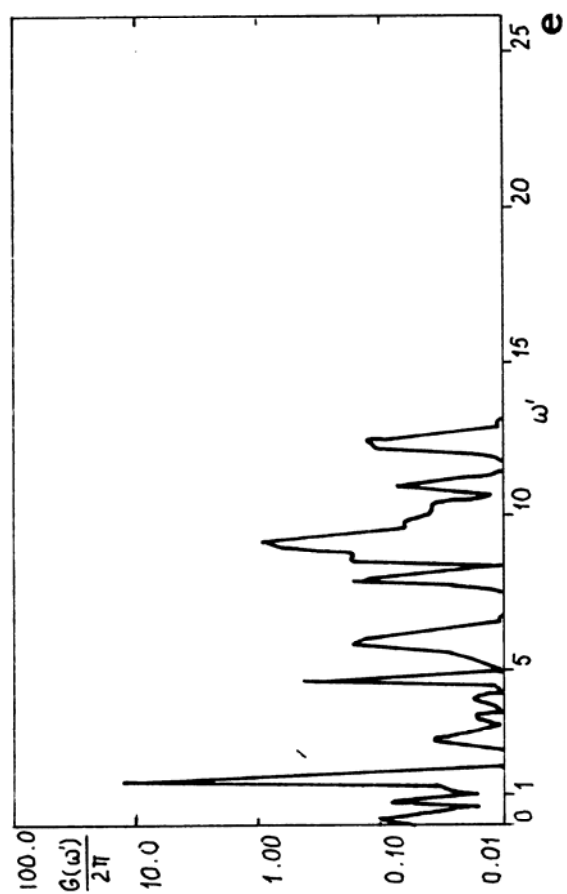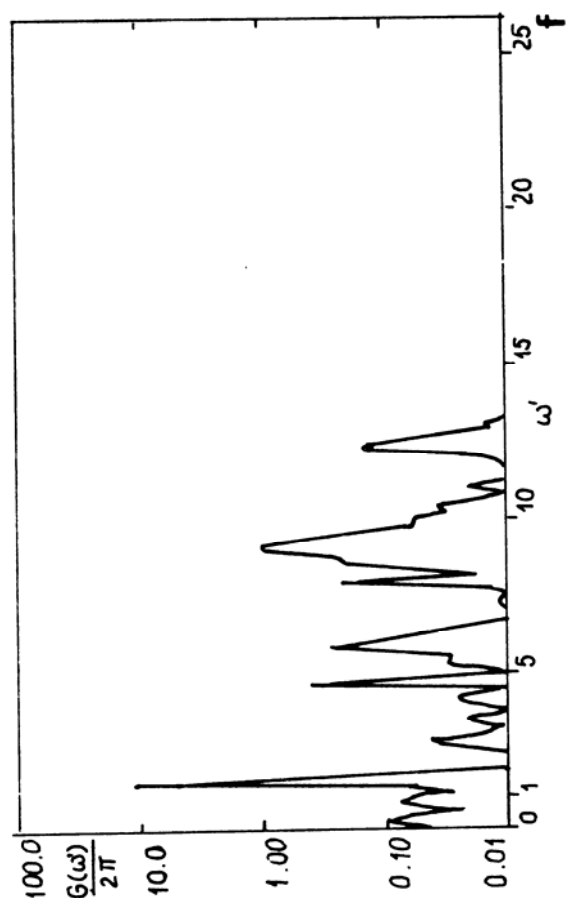
Fig. 2

Fig. 2. Frequency spectra for the parameters as in Fig. 1

new components in the frequency spectra appear and the spectrum in some narrow ranges looks like broad band — the motion exhibits small chaos (see Figs. 1d and 2d). With further decrease of α chaos becomes more profound. The strange attractors are presented on Poincarè's maps (Figs. 1e, f, g) and the adequate frequency spectra are shown in Fig. 2e, f, g. The investigated equation, generally, is particularly sensitive for the changes of the time delay value τ_0. The increase of this value causes also an increase of the magnitude of the strange attractor (Fourier spectra become in this case more broad band), while for the decrease of τ_0 the regular motion appears.

4. Concluding Remarks

The paper presents a procedure leading to the detection of chaotic motion in the Van der Pol-Duffing's oscillator with delay. It consists in such a choice of parameters of the analysed oscillator, that Hopf bifurcation of the periodic motion will occur. This motion has been approximated using one basic harmonic and then the critical value of α for the Hopf bifurcation of this steady state has been obtained. In this case $\alpha_c = .09$.

The numerical results show that the bifurcation takes place for $\alpha = .1$ (near the previously analytically obtained value α_c) and the transition from the regular to irregular motion can be observed with the decrease of the dimensionless damping α. Chaos increases with the decrease of this coefficient and the frequency spectrum becoming broad band occurs in a wide range around the previously occurring discrete values.

Acknowledgement

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